

Appreciating Generating Functions as a Bridge between Discrete and Continuous Mathematics

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Abstract

Generating functions transform an infinite sequence into the coefficients of a power series. It enables us to approach problem of sequence using the methods we have for algebraic problems. Given any power series, we treat this power series as an algebraic object where we have no concern in the convergence of the series. In this paper, we shall develop a generalized technique for constructing the generating functions of some algebraic functions, proper rational functions and binomial functions (both positive and negative powers) using method of differentiation. Also, we shall apply the techniques generalized in obtaining series expansion of some functions.

Keywords: Taylor series, Power series, proper rational function, differentiable function and Sequence of numbers

1.1 INTRODUCTION

In mathematics, the generating function technique provides a way of using a single function to encode an infinite sequence of numbers by treating them as the coefficients of a formal power series. Thus, this series is called the generating function of the sequence (Solomentsev, 2001). However, generating function technique provides a natural and elegant ways of dealing with sequence of numbers by associating a function of a continuous variable with a sequence. In this way, generating functions provide a bridge between continuous and discrete mathematics. Unlike an ordinary series, the formal power series is not required to converge (Haukkanen, 2019). Abraham De Moivre was the first person who introduced generating functions in 1730 in order to solve general problems of linear recurrence. James Stirling in 1730 then extended this theory in his Methodus Differentialis by using integration and differentiation to solve the same problems. However, Leonhard Paul Euler in 1747 began to extend their use to new field in mathematics such as combinatorics. Also, in 1812, Pierre-Simon De Laplace took the technique into the field of probability theory in his work “Théorie Analytique des Probabilités” (Vilenkin, 1969). Many other researchers after them have developed a lot techniques in generating functions. One can generalize to formal power series in more than one indeterminate, to encode information about infinite multi-dimensional arrays of numbers. Sometimes, we view generating

function in the perspective of the formal power series (Flajolet, and Sedgewick, 2009). Thus, questions of convergence may arise and analytic techniques would then come to play an important role. The major idea of generating function is, instead of an infinite sequence of numbers, we search for a single function which encodes the sequence of the numbers but not a function that gives n th term as output. Rather, a function whose power series shows the terms of the sequence. Generating functions are employed while solving many counting problems as questions about the convergence of these series are ignored (Michaels, 2013 and Comtet, 1974).

There are many types of generating functions which include exponential generating functions, ordinary generating functions, Bell Series, Dirichlet series and Lambert series. They are used to find statistical properties, average, recurrence formulas, prove identities and determine features of sequence. Every sequence has a generating function of each type but the way they are handled varies except Dirichlet and Lambert series which requires its indices to start with one rather than zero. Generating functions are not functions in the formal sense of mapping from domain to co-domain. Sometimes, generating functions are called generating series because the terms of the series can be said to be the generator of its sequence of term coefficients (Lando, 2003). Sometimes, we express generating functions in a closed form instead of as a series (Comtet, 1974).

In this paper, we shall develop a generalized technique for constructing the generating functions of some algebraic functions, proper rational functions and binomial functions (both positive and negative powers) using method of differentiation. Also, we shall apply the techniques generalized in obtaining series expansion of some functions.

2.1 BASIC DEFINITIONS

2.1.1 SEQUENCE AND SERIES

A sequence of numbers is the collection of numbers $a_1, a_2, a_3, \dots, a_n, \dots$ which is written in order. Here, a_1 is called the first term, a_2 the second term, a_3 the third term, $\dots, a_n$ the n th term, $\dots$. Sometimes, there is a recursive relationship between a term and the succeeding or preceding terms. A series is then the infinite sum of the terms of a sequence (Obiajulu and Ubaoji, 2017)

2.1.2 POWER SERIES

A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Where n is positive integer and a_n is the coefficient of x^n . Power series can also be defined in combinatorics as generating functions (Stroud and Dexter, 2007; Wilf, 1994)

2.1.3 CONVERGENT SERIES

Let $s = \sum_{k \geq 0} a_k$ be a series. The sequence $\{S_n\}$ given by $S_n = \sum_{k=0}^n a_k$ is called the sequence of the partial sum of the series. A series s converges if and only if the sequence $\{S_n\}$ of partial sum of the series converges. In other words, a series converges if the limit of the series exist (Gessel, 2013; Stroud and Dexter, 2007)

2.1.4 ORDINARY GENERATING FUNCTIONS

It is a generating function which is of the form

$$B(x) = \sum_{n=0}^{\infty} f_n x^n = f_0 + f_1 x + f_2 x^2 + \dots$$

Where f_n is the constant coefficient of the series (Skoch, 2015; Stroud and Dexter, 2007)

2.1.5 EXPONENTIAL GENERATING FUNCTION

Exponential generating function of the sequence $\{f_n\}_{n \geq 0}$ is any formal power series that is of the form

$$B(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}$$

Where $B(x)$ is the exponential generating function of $f_0, f_1, f_2, f_3, f_4, \dots$ (Stroud and Dexter, 2007)

2.1.6 COMBINATORICS

It is a mathematical area that deals with problems of arrangement, selection and operations within a finite set in accordance with some constraints (Prasanna, 2006 and Wilf, 1994).

2.1.7 GENERATING FUNCTION OF A GEOMETRIC SEQUENCE

Given any geometric sequence a^n , its generating function is

$$g(x) = \sum_{n=0}^{\infty} a^n x^n = \frac{1}{1 - ax}$$

(Stroud and Dexter, 2007).

2.1.8 RATIONAL FUNCTION

Let $p(x)$ and $q(x)$ be polynomials in a variable x . $\frac{p(x)}{q(x)}$ is called a rational function in x provided that $q(x) \neq 0$ since division by zero is not defined. Thus, $\frac{p(x)}{q(x)}$ is defined only for values of x where $q(x) \neq 0$. If the degree of $p(x)$ is less than the degree of $q(x)$, then the rational function is called a proper rational function (Obiajulu & Ubaoji, 2017).

2.1.9 DIFFERENTIABLE FUNCTION

A function $f(x)$ is said to be differentiable at x if

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ exists (Obiajulu \& Ubaoji, 2017).}$$

3.1 IMPORTANT THEOREMS

3.1.1 TAYLOR SERIES THEOREM

The Taylor series of complex or real valued function $f(x)$ which is infinitely differentiable at complex or real number a is the power series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

Where $f^n(a)$ is the n th derivative of f at $x = a$. When $a = 0$ in the Taylor series, the series can also be called Maclaurin series (Stroud and Dexter, 2007)

3.1.2 NEWTON'S BINOMIAL THEOREM

The binomial theorem deals with the expansion of any expression of this kind $(m+n)^p$. It states that for any non-negative integer p , we have

$$(m+n)^p = \sum_{r=0}^p \binom{p}{r} m^r n^{p-r}$$

Where $\binom{p}{r} = \frac{p!}{(p-r)!r!}$ is the coefficient of the binomials (Stroud and Dexter, 2007 and Prasanna, 2006).

3.1.3 THEOREM

For any positive integers n and r , we have

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r} \quad (\text{Prasanna, 2006})$$

3.1.4 AUTOMATIC GENERATOR FOR GENERATING FUNCTIONS OF ALGEBRAIC FUNCTIONS

For the function

$$f(x) = \frac{a}{1-rx} \quad \text{for } r > 0,$$

its generating function is

$$g(x) = a \sum_{k \geq 0} r^k x^k$$

3.1.5 AUTOMATIC GENERATOR FOR GENERATING FUNCTIONS OF PROPER RATIONAL FUNCTIONS

For the function

$$f(x) = \frac{a}{(1-bx)^m} \quad \text{for } b > 0 \text{ and } m > 0,$$

its generating function is

$$g(x) = a \sum_{r \geq 0} \binom{m-1+r}{r} b^r x^r$$

3.1.6 AUTOMATIC GENERATOR FOR GENERATING FUNCTIONS OF BINOMIAL FUNCTION (BOTH POSITIVE AND NEGATIVE POWERS)

If a function $f(x) = c(a+bx)^m$, then its generating function is

$$g(x) = \begin{cases} c \sum_{r=0}^m \binom{m}{r} a^{m-r} b^r x^r; & \text{if } b > 0, m > 0 \\ c \sum_{r=0}^m \binom{m}{r} (-1)^r a^{m-r} d^r x^r; & \text{if } b = -d < 0, m > 0 \\ c \sum_{r \geq 0} \binom{-m-1+r}{r} (-1)^r a^{-(m+r)} b^r x^r; & \text{if } b > 0, m < 0 \\ c \sum_{r \geq 0} \binom{-m-1+r}{r} a^{-(m+r)} d^r x^r; & \text{if } b = -d < 0, m < 0 \end{cases}$$

4.1 SPECIAL CASE OF OUR RESEARCH RESULTS

RESEARCH RESULT 4.1.1: Develop a generalized technique for constructing the generating functions of algebraic function

$$f(x) = \frac{a}{1-rx}$$

Solution

$$f(x) = \frac{a}{1-rx} = a(1-rx)^{-1}$$

Using differentiation technique, we have

$$f'(x) = ar(1-rx)^{-2}$$

$$f''(x) = 2! ar^2(1-rx)^{-3}$$

$$f'''(x) = 3! ar^3(1-rx)^{-4}$$

⋮

$$f^k(x) = k! ar^k(1-rx)^{-(k+1)}$$

If $x = 0$, we have

$$f^k(0) = k! ar^k$$

Now, make ar^k the subject of formula and apply the notion of generating functions, we have

$$g(x) = \sum_{k \geq 0} \frac{f^k(0)}{k!} x^k = \sum_{k \geq 0} \frac{k! a r^k}{k!} x^k = \sum_{k \geq 0} a r^k x^k = a \sum_{k \geq 0} r^k x^k$$

Thus, the generalized technique for obtaining the generating functions of algebraic function

$$f(x) = \frac{a}{1 - rx}$$

is

$$g(x) = a \sum_{k \geq 0} r^k x^k$$

EXAMPLE 4.1.1.1

Obtain the series expansion of the function

$$f(x) = \frac{1}{1 - x}$$

Solution

We compare the function $f(x) = \frac{1}{1-x}$ with $\frac{a}{1-rx}$, we have

$$\frac{1}{1-x} = \frac{a}{1-rx} \cdot \quad \text{Here, } a = 1 \text{ and } r = 1$$

Thus, using our generalized technique for obtaining the generating functions of algebraic function, we have

$$g(x) = a \sum_{k \geq 0} r^k x^k$$

$$\therefore g(x) = 1[1^0 x^0 + 1^1 x^1 + 1^2 x^2 + 1^3 x^3 + 1^4 x^4 + \dots]$$

$$g(x) = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\text{Thus, the series is } g(x) = 1 + x + x^2 + x^3 + x^4 + \dots$$

Example 4.1.1. 2

Find the series expansion of the function

$$f(x) = \frac{3}{1-3x}$$

Solution

We compare the function $f(x) = \frac{3}{1-3x}$ with $\frac{a}{1-rx}$, we have

$$\frac{3}{1-3x} = \frac{a}{1-rx} \quad \text{Here, } a = 3 \text{ and } r = 3$$

Thus, using our generalized technique for obtaining the generating functions of algebraic function, we have

$$g(x) = a \sum_{k \geq 0} r^k x^k$$

$$\therefore g(x) = 3(1 + 3x + 9x^2 + 27x^3 + 81x^4 + \dots)$$

$$= 3 + 9x + 27x^2 + 81x^3 + 243x^4 + \dots$$

$$\text{Thus, the series is } g(x) = 3 + 9x + 27x^2 + 81x^3 + 243x^4 + \dots$$

EXAMPLE 4.1.1.3

Obtain the series expansion of the function

$$f(x) = \frac{4x}{1-5x}$$

Solution

We compare $f(x) = \frac{4x}{1-5x}$ with $\frac{a}{1-rx}$, we have

$$\frac{4x}{1-5x} = \frac{a}{1-rx} \quad \{ \text{Here, } a = 4x \text{ and } r = 5 \}$$

Thus, using our generalized technique for obtaining the generating functions of algebraic function, we have

$$g(x) = a \sum_{k \geq 0} r^k x^k$$

$$\therefore g(x) = 4x[5^0 x^0 + 5^1 x^1 + 5^2 x^2 + 5^3 x^3 + 5^4 x^4 + \dots]$$

$$g(x) = 4x[1 + 5x + 25x^2 + 125x^3 + 625x^4 + \dots]$$

$$g(x) = 4x + 20x^2 + 100x^3 + 500x^4 + 2500x^5 + \dots$$

Thus, the series is $g(x) = 4x + 20x^2 + 100x^3 + 500x^4 + 2500x^5 + \dots$

EXAMPLE 4.1.1.4

Find the coefficient of x^6 in the expansion of the function

$$f(x) = \frac{2}{1-4x}$$

Solution

We compare the function $f(x) = \frac{2}{1-4x}$ with $\frac{a}{1-rx}$ and we have

$$\frac{2}{1-4x} = \frac{a}{1-rx} . \quad \text{Here, } a = 2 \text{ and } r = 4$$

Thus, using our generalized technique for obtaining the generating functions of algebraic function, we have

$$g(x) = a \sum_{k \geq 0} r^k x^k$$

$$\therefore g(x) = 2(4^0 x^0 + 4^1 x^1 + 4^2 x^2 + 4^3 x^3 + 4^4 x^4 + 4^5 x^5 + 4^6 x^6 + \dots)$$

$$g(x) = 2(1 + 4x + 16x^2 + 64x^3 + 256x^4 + 1024x^5 + 4096x^6 + \dots)$$

$$g(x) = 2 + 8x + 32x^2 + 128x^3 + 512x^4 + 2048x^5 + 8192x^6 + \dots$$

From the series above, we can see that the coefficient of x^6 is 8192

RESEARCH RESULT 4.1.2: Develop a generalized technique for constructing the generating functions of proper rational functions

$$f(x) = \frac{a}{(1 - bx)^m}$$

Solution

$$f(x) = \frac{a}{(1 - bx)^m} = a(1 - bx)^{-m}$$

Using differentiation method, we have

$$f'(x) = amb(1 - bx)^{-(m+1)}$$

$$f''(x) = amb^2(m + 1)(1 - bx)^{-(m+2)}$$

$$f'''(x) = amb^3(m + 1)(m + 2)(1 - bx)^{-(m+3)}$$

⋮

$$f^r(x) = amb^r(m + 1)(m + 2)(m + 3) \dots (1 - bx)^{-(m+r)}$$

$$f^r(0) = amb^r(m + 1)(m + 2)(m + 3) \dots$$

Transforming the above illustration into the notion of generating functions, we have

$$g(x) = a \sum_{r \geq 0} \binom{m - 1 + r}{r} b^r x^r$$

Thus, the generalized technique for obtaining the generating functions of proper rational function

$$f(x) = \frac{a}{(1 - bx)^m}$$

is

$$g(x) = a \sum_{r \geq 0} \binom{m - 1 + r}{r} b^r x^r$$

EXAMPLE 4.1.2.1

Find the series expansion of the function

$$f(x) = \frac{1}{(1-x)^2}$$

Solution

We compare the function $f(x) = \frac{1}{(1-x)^2}$ with $\frac{a}{(1-bx)^m}$, we have

$$\frac{1}{(1-x)^2} = \frac{a}{(1-bx)^m} . \quad \text{Here, } a = 1, b = 1 \text{ and } m = 2$$

Thus, using our generalized technique for obtaining the generating functions of proper rational function, we have

$$\begin{aligned} g(x) &= a \sum_{r \geq 0} \binom{m-1+r}{r} b^r x^r \\ \therefore g(x) &= 1 \binom{2-1+0}{0} 1^0 x^0 + 1 \binom{2-1+1}{1} 1^1 x^1 + 1 \binom{2-1+2}{2} 1^2 x^2 \\ &\quad + 1 \binom{2-1+3}{3} 1^3 x^3 + \dots \\ g(x) &= 1 + 2x + 3x^2 + 4x^3 + \dots \end{aligned}$$

Thus, the series is $g(x) = 1 + 2x + 3x^2 + 4x^3 + \dots$

EXAMPLE 4.1.2.2

Find the series expansion of the function

$$f(x) = \frac{2}{(1-3x)^4}$$

Solution

Compare the function $f(x) = \frac{2}{(1-3x)^4}$ with $\frac{a}{(1-bx)^m}$ we have

$$\frac{2}{(1-3x)^4} = \frac{a}{(1-bx)^m} . \quad \text{Here, } a = 2, b = 3 \text{ and } m = 4$$

Thus, using our generalized technique for obtaining the generating functions of proper rational function, we have

$$g(x) = a \sum_{r \geq 0} \binom{m-1+r}{r} b^r x^r$$

$$\therefore g(x) = 2 \binom{4-1+0}{0} 3^0 x^0 + 2 \binom{4-1+1}{1} 3^1 x^1 + 2 \binom{4-1+2}{2} 3^2 x^2 + 2 \binom{4-1+3}{3} 3^3 x^3 + \dots$$

$$g(x) = 2 \binom{3}{0} 3^0 x^0 + 2 \binom{4}{1} 3^1 x^1 + 2 \binom{5}{2} 3^2 x^2 + 2 \binom{6}{3} 3^3 x^3 + \dots$$

$$g(x) = 2(1)3^0 x^0 + 2(4)3^1 x^1 + 2(10)3^2 x^2 + 2(20)3^3 x^3 + \dots$$

$$g(x) = 2 + 24x + 180x^2 + 1080x^3 + \dots$$

Thus, the series is $g(x) = 2 + 24x + 180x^2 + 1080x^3 + \dots$

Example 4.1.2.3

Find the series expansion of the function

$$f(x) = \frac{3x}{(1-4x)^5}$$

Solution

We compare $f(x) = \frac{3x}{(1-4x)^5}$ with $\frac{a}{(1-bx)^m}$, we have

$$f(x) = \frac{3x}{(1-4x)^5} = \frac{a}{(1-bx)^m} \quad \{Here, a = 3x, b = 4 \text{ and } m = 5\}$$

Thus, using our generalized technique for obtaining the generating functions of proper rational function, we have

$$g(x) = a \sum_{r \geq 0} \binom{m-1+r}{r} b^r x^r$$

$$\therefore g(x) = 3x \left[\binom{5-1+0}{0} 4^0 x^0 + \binom{5-1+1}{1} 4^1 x^1 + \binom{5-1+2}{2} 4^2 x^2 + \binom{5-1+3}{3} 4^3 x^3 + \binom{5-1+4}{4} 4^4 x^4 + \dots \right]$$

$$g(x) = 3x \left[\binom{4}{0} 4^0 x^0 + \binom{5}{1} 4^1 x^1 + \binom{6}{2} 4^2 x^2 + \binom{7}{3} 4^3 x^3 + \binom{8}{4} 4^4 x^4 + \dots \right]$$

$$g(x) = 3x[1 + 5(4x) + 15(16x^2) + 35(64x^3) + 70(256x^4) + \dots]$$

$$g(x) = 3x[1 + 20x + 240x^2 + 2240x^3 + 17920x^4 + \dots]$$

$$g(x) = 3x + 60x^2 + 720x^3 + 6720x^4 + 53760x^5 + \dots$$

Thus, the series is $g(x) = 3x + 60x^2 + 720x^3 + 6720x^4 + 53760x^5 + \dots$

EXAMPLE 4.1.2.4

Find the series expansion of the function

$$f(x) = \frac{1+x}{(1-x)^3}$$

Solution

We compare $f(x) = \frac{1+x}{(1-x)^3}$ with $\frac{a}{(1-bx)^m}$, we have

$$f(x) = \frac{1+x}{(1-x)^3} = \frac{a}{(1-bx)^m} \quad \{ \text{Here, } a = (1+x), b = 1 \text{ and } m = 3 \}$$

Thus, using our generalized technique for obtaining the generating functions of proper rational function, we have

$$g(x) = a \sum_{r \geq 0} \binom{m-1+r}{r} b^r x^r$$

$$\therefore g(x) = (1+x) \left[\binom{3-1+0}{0} 1^0 x^0 + \binom{3-1+1}{1} 1^1 x^1 + \binom{3-1+2}{2} 1^2 x^2 + \binom{3-1+3}{3} 1^3 x^3 + \dots \right]$$

$$g(x) = (1+x) \left[\binom{2}{0} 1^0 x^0 + \binom{3}{1} 1^1 x^1 + \binom{4}{2} 1^2 x^2 + \binom{5}{3} 1^3 x^3 + \dots \right]$$

$$g(x) = (1+x)[1 + 3x + 6x^2 + 10x^3 + \dots]$$

$$g(x) = 1 + 4x + 9x^2 + 16x^3 + \dots$$

Thus, the series is $g(x) = 1 + 4x + 9x^2 + 16x^3 + \dots$

EXAMPLE 4.1.2.5

Find the coefficient of x^5 in the expansion of the function

$$g(x) = \frac{3}{(1-5x)^2}$$

Solution.

We compare the function $f(x) = \frac{3}{(1-5x)^2}$ with $\frac{a}{(1-bx)^m}$, we have

$$\frac{3}{(1-5x)^2} = \frac{a}{(1-bx)^m} \cdot \quad \text{Here, } a = 3, b = 5 \text{ and } m = 2$$

Thus, using our generalized technique for obtaining the generating functions of proper rational function, we have

$$g(x) = a \sum_{r \geq 0} \binom{m-1+r}{r} b^r x^r$$

$$\begin{aligned} \therefore g(x) &= 3 \binom{2-1+0}{0} 5^0 x^0 + 3 \binom{2-1+1}{1} 5^1 x^1 + 3 \binom{2-1+2}{2} 5^2 x^2 \\ &\quad + 3 \binom{2-1+3}{3} 5^3 x^3 + 3 \binom{2-1+4}{4} 5^4 x^4 + 3 \binom{2-1+5}{5} 5^5 x^5 + \dots \end{aligned}$$

$$\begin{aligned} g(x) &= 3 \binom{1}{0} 5^0 x^0 + 3 \binom{2}{1} 5^1 x^1 + 3 \binom{3}{2} 5^2 x^2 + 3 \binom{4}{3} 5^3 x^3 + 3 \binom{5}{4} 5^4 x^4 + 3 \binom{6}{5} 5^5 x^5 + \dots \\ &= 3 + 30x + 225x^2 + 1500x^3 + 9375x^4 + 56250x^5 + \dots \end{aligned}$$

From the series above, we can see that the coefficient of x^5 is 56250.

RESEARCH RESULT 4.1.3: Develop a generalized technique for constructing the generating functions of binomial function with positive power

$$f(x) = a(b+cx)^m$$

Solution

$$f(x) = a(b+cx)^m$$

Using differentiation method, we have

$$f'(x) = mac(b+cx)^{m-1}$$

$$f''(x) = m(m-1)ac^2(b+cx)^{m-2}$$

$$f'''(x) = m(m-2)(m-1)ac^3(b+cx)^{m-3}$$

⋮

$$f^{(r)}(x) = m_{pr}ac^r(b+cx)^{m-r}; \quad f^{(r)}(0) = m_{pr}ac^rb^{m-r}$$

Transforming the above illustration into the notion of generating functions, we have

$$g(x) = \sum_{r=0}^m \frac{f^{(r)}(0)ac^rb^{m-r}}{r!} x^r = a \sum_{r=0}^m \binom{m}{r} b^{m-r} c^r x^r$$

Thus, the generalized technique for obtaining the generating functions of binomial function with positive power

$$f(x) = a(b+cx)^m$$

is

$$g(x) = a \sum_{r=0}^m \binom{m}{r} b^{m-r} c^r x^r$$

EXAMPLE 4.1.3.1

Obtain the expansion of the function

$$f(x) = 3(2+4x)^5$$

Solution

Using our generalized technique, we compare the function $f(x) = 3(2+4x)^5$ with $a(b+cx)^m$, we have

$$f(x) = 3(2+4x)^5 = a(b+cx)^m \quad \{ \text{Here, } a = 3, b = 2, c = 4 \text{ and } m = 5 \}$$

Thus, using our generalized technique for obtaining the generating functions of binomial function with positive power, we have

$$g(x) = a \sum_{r=0}^m \binom{m}{r} b^{m-r} c^r x^r$$

$$\therefore g(x) = 3 \left[\binom{5}{0} 2^{5-0} 4^0 x^0 + \binom{5}{1} 2^{5-1} 4^1 x^1 + \binom{5}{2} 2^{5-2} 4^2 x^2 + \binom{5}{3} 2^{5-3} 4^3 x^3 + \binom{5}{4} 2^{5-4} 4^4 x^4 + \binom{5}{5} 2^{5-5} 4^5 x^5 \right]$$

$$g(x) = 3[1(32) + 5(16)4x + 10(8)16x^2 + 10(4)64x^3 + 5(2)256x^4 + 1(1)1024x^5]$$

$$g(x) = 3[32 + 320x + 1280x^2 + 2560x^3 + 2560x^4 + 1024x^5]$$

$$g(x) = 96 + 960x + 3840x^2 + 7680x^3 + 7680x^4 + 3072x^5$$

Thus, the expansion of $f(x) = 3(2 + 4x)^5$ is

$$g(x) = 96 + 960x + 3840x^2 + 7680x^3 + 7680x^4 + 3072x^5$$

RESEARCH RESULT 4.1.4: Develop a generalized technique for constructing the generating functions of binomial function with negative power

$$f(x) = a(b + cx)^{-m}$$

Solution

$$f(x) = a(b + cx)^{-m}$$

Using differentiation method, we have

$$f'(x) = -mac(b + cx)^{-m-1} = -mac(b + cx)^{-(m+1)}$$

$$f''(x) = m(m+1)ac^2(b + cx)^{-(m+2)}$$

$$f'''(x) = -m(m+2)(m+1)ac^3(b + cx)^{-(m+3)}$$

$$f^{iv}(x) = m(m+3)(m+2)(m+1)ac^4(b + cx)^{-(m+4)}$$

⋮

$$f^{(r)}(x) = (-1)^r m_{pr} ac^r (b + cx)^{-(m+r)}; \quad f^{(r)}(0) = (-1)^r m_{pr} ac^r b^{-(m+r)}$$

Transforming the above illustration into the notion of generating functions, we have

$$g(x) = \sum_{r \geq 0} \frac{f^{(r)}(0)(-1)^r a c^r b^{-(m+r)}}{r!} x^r = a \sum_{r \geq 0} \binom{-m-1+r}{r} (-1)^r b^{-(m+r)} c^r x^r$$

Thus, the generalized technique for obtaining the generating functions of binomial function with negative power

$$f(x) = a(b + cx)^{-m}$$

is

$$g(x) = a \sum_{r \geq 0} \binom{-m-1+r}{r} (-1)^r b^{-(m+r)} c^r x^r$$

EXAMPLE 4.1.4.1

Find the series expansion of the function

$$f(x) = 2(1 - 3x)^{-4}$$

Solution

Using our generalized technique for obtaining generating functions of binomial function with negative power, we have

$$g(x) = a \sum_{r \geq 0} \binom{-m-1+r}{r} (-1)^r b^{-(m+r)} c^r x^r$$

To test the efficiency of the above generalized technique, we compare $f(x) = 2(1 - 3x)^{-4}$ with $a(b + cx)^m$, we have

$$2(1 - 3x)^{-4} = a(b + cx)^m$$

$$\text{Here, } a = 2, b = 1, c = -3 \text{ and } m = -4$$

$$\begin{aligned} \therefore g(x) = 2 & \left[\binom{-(-4)-1+0}{0} (-1)^0 (1)^{-(-4+0)} (-3)^0 x^0 \right. \\ & + \binom{-(-4)-1+1}{1} (-1)^1 (1)^{-(-4+1)} (-3)^1 x^1 \\ & + \binom{-(-4)-1+2}{2} (-1)^2 (1)^{-(-4+2)} (-3)^2 x^2 \\ & \left. + \binom{-(-4)-1+3}{3} (-1)^3 (1)^{-(-4+3)} (-3)^3 x^3 + \dots \right] \end{aligned}$$

$$g(x) = 2 \left[\binom{3}{0} (-3)^0 x^0 + \binom{4}{1} (-1)(-3)^1 x^1 + \binom{5}{2} (-1)^2 (-3)^2 x^2 + \binom{6}{3} (-1)^3 (-3)^3 x^3 + \dots \right]$$

$$g(x) = 2[1 + 4(-1)(-3)x + 10(-1)9x^2 + 20(-1)(-27)x^3 + \dots]$$

$$g(x) = 2[1 + 12x + 90x^2 + 540x^3 + \dots]$$

$$g(x) = 2 + 24x + 180x^2 + 1080x^3 + \dots$$

Thus, the series is $g(x) = 2 + 24x + 180x^2 + 1080x^3 + \dots$

5.1 CONCLUSION

Generating function plays an important role in the analysis of discrete problems in mathematics with function. Its beauty can be appreciated because it can be used to manipulate many complicated mathematical problems with very little effort as question about convergence are considered a non-issue. In this study, we developed generalized techniques for constructing the generating functions of some algebraic functions, proper rational functions and binomial functions (both positive and negative powers) using method of differentiation. Also, we applied the techniques generalized in obtaining series expansion of some functions.

6.1 RECOMMENDATION

Further research should be done on generating function so that it can be used to manipulate mathematical problems in Fourier series and Laurent series.

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