

Building Generating Functions for the Sequences $f_{m+1} = 2(f_m + m^k)$

¹Moore Chika.; ²J. E. Okeke and ³E. I. Chukwuma

¹Department of Mathematics, Nnamdi Azikiwe University, Awka, Anambra State

^{2,3}Department of Mathematics, Chukwuemeka Odumegwu Ojukwu University, Uli, Anambra State

Email

¹profchikamoore@gmail.com

²justina.okeke@gmail.com

³symbol4maths@gmail.com

Abstract

Generating functions provide a bridge between continuous and discrete mathematics. We use generating functions to obtain recursive sequences and turn them into closed form function. This allows us to solve many complicated counting problems with very little effort. In this paper, we shall develop generalized techniques for constructing the generating functions of sequences of the form $f_{m+1} = 2(f_m + m^k)$ for all $m \geq 0, k > 0$ and $f_0 = 0$ using method of differencing in generating function. Also, we shall apply the technique of generating functions in solving some important problems of recurrence relations.

Keywords: Sequence, polynomial function, differencing in generating function, ordinary generating function and exponential generating function.

1.1 INTRODUCTION

Generating functions are widely used and can be regarded as among the powerful tools of mathematical analysis, combinatorics, statistics etc. The approach of generating function is easily used for tackling combinatorial problems. Some of the earliest researchers who applied the approach of generating functions to solve recurrence problems include Abraham De Moivre in 1730 and Leonhard Paul Euler in 1747, they extended generating functions' approach by applying it to solve general problems of linear recurrence, partition problems and its equivalents (Stillwell, 2010). Many researchers such as Michael, 2022; Sergio, 2021; Ritu, 2020; Peter, 2019; Stanley, 2012; Flajolet and Sedgewick, 2009; Egorychev, 2008; Lando, 2003; Wilf, 1994; Riordan, 1980; Comtet, 1974 and a lot of others have made valuable contributions to the development of the generating functions' approach and its applications (Vuk and Nicola, 2022).

Furthermore, generating functions are the main ways of explaining polynomials without any difficulties. Polynomials serve as the elementary mathematical objects which can be applied in various fields of applied and pure mathematics (Hari, 2022). With regards to this, Generating functions permit us to solve problems of sequence with the method we used while solving problems in algebra. Thus, when given power series to handle, we treat the power series in algebraic way where we have no concern whether the series converges or not (Dan and Mark, 2019; Flajolet, and Sedgewick, 2009). Generating functions provide a bridge between continuous

and discrete mathematics. This allows us to solve many complicated counting problems with very little effort.

In this paper, we shall develop generalized techniques for constructing the generating functions of sequences of the form $f_{m+1} = 2(f_m + m^k)$ for all $m \geq 0, k > 0$ and $f_0 = 0$ using method of differencing in generating function. Also, we shall apply the technique of generating functions in solving problems of recurrence relations.

2.1 BASIC DEFINITIONS

2.1.1 SEQUENCE AND SERIES

A sequence of numbers is the collection of numbers $a_1, a_2, a_3, \dots, a_n, \dots$ which is written in order. Here, a_1 is called the first term, a_2 the second term, a_3 the third term, $\dots, a_n$ the n th term, $\dots$. Sometimes, there is a recursive relationship between a term and the succeeding or preceding terms. A series is then the infinite sum of the terms of a sequence (Obiajulu and Ubaoji, 2017).

2.1.2 POWER SERIES

A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Where n is positive integer and a_n is the coefficient of x^n . Power series can also be defined in combinatorics as generating functions (Amartya, 2022; Stroud and Dexter, 2007; Wilf, 1994)

2.1.3 CONVERGENT SERIES

Let $s = \sum_{k \geq 0} a_k$ be a series. The sequence $\{S_n\}$ given by $S_n = \sum_{k=0}^n a_k$ is called the sequence of the partial sum of the series. A series s converges if and only if the sequence $\{S_n\}$ of partial sum of the series converges. In other words, a series converges if the limit of the series exist (Amartya, 2022; Gessel, 2013; Stroud and Dexter, 2007)

2.1.4 ORDINARY GENERATING FUNCTIONS

It is a generating function of the form

$$B(x) = \sum_{n=0}^{\infty} f_n x^n = f_0 + f_1 x + f_2 x^2 + \dots \quad (2.1)$$

Where f_n is the coefficient of the series (Skoch, 2015; Stroud and Dexter, 2007)

2.1.5 EXPONENTIAL GENERATING FUNCTION

Exponential generating function of the sequence $\{f_n\}_{n \geq 0}$ is any formal power series that is of the form

$$B(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}$$

Where $B(x)$ is the exponential generating function of $f_0, f_1, f_2, f_3, f_4, \dots$ (Stroud and Dexter, 2007)

2.1.6 COMBINATORICS

It is a mathematical area that deals with problems of arrangement, selection and operations within a finite set in accordance with some constraints (Weisstein, 2017 and Wilf, 1994).

2.1.7 NATURAL NUMBERS “N”

Natural numbers 1, 2, 3, 4, ... are positive or non-negative integers. They can also be regarded as counting members of a set. The set of natural numbers is closed under the operations of multiplication and addition (Stroud and Dexter, 2007).

2.1.8 CLOSED FORM

In constructing generating functions, it is interesting to determine if the generating function has a closed form expression or an expression for the coefficient of the general term. Thus, the expression

$$g(x) = \sum_{n \geq 0} a^n x^n \text{ is not in a closed form whereas,}$$

$g(x) = \frac{1}{1-ax}$, an equivalent expression to some extent, is a closed form (Weisstein, 2017).

2.1.9 DIFFERENCING IN GENERATING FUNCTION

Generating functions can be gotten using the method of multiplication (by x or *constant*), addition, substitution and differentiation. To use each of these, one must know how to transform the sequence into any sequence of choice. One thing we always considered is the sequence of differences between terms of a sequence. This as well will turn out to be helpful in finding the generating functions. Thus, the sequence of differencing is often simpler than the original sequence. Thus, if we know a generating function for the differences, we would like to use this to find generating function for the original sequence. For example; given 1, 3, 5, 7, 9, ..., we can use differencing to obtain the generating functions as follows;

$$\text{Let } g(x) = A = 1 + 3x + 5x^2 + 7x^3 + 9x^4 + \dots$$

$$- \quad xA = 0 + x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \dots$$

$$A(1 - x) = 1 + 2x + 2x^2 + 2x^3 + 2x^4 + \dots$$

$$A(1 - x) = 1 + \left(\frac{2x}{1 - x} \right)$$

$$A = g(x) = \frac{1 + x}{(1 - x)^2}$$

The method of differencing in generating function is, in any given infinite sequence of numbers, it searches for a single function to encode the infinite sequence of numbers (Lando, 2003).

2.1.10 POLYNOMIAL FUNCTION

A polynomial function in x is a function which contains only a finite number of positive integer powers of x , usually arranged in the order of ascending or descending powers. The degree of a Polynomial is the highest power of the variable occurring in the function; for instance $f(x) =$

$5x^4 + 7x^3 + 2x^2 + 3x + 1$ is a 4th degree polynomial. Thus, the general form of a polynomial function can be seen in the form of power series like $f(x) = \sum_{m=0}^{\infty} f_m x^m$ provided that there exist $n_0 \in \mathbb{N}$ such that $f_m = 0$ for all $m > n_0$ (Stroud and Dexter, 2007)

3.1 IMPORTANT THEOREMS

3.1.1 THEOREM OF GENERATING FUNCTION

Let the generating function G be given by

$$G(x) = \sum_{n \geq 0} g_n \frac{x^n}{n!} \quad \text{where } g_n \text{ is the } n\text{th fibonacci number.}$$

Show that the differential equation $G''(x) = G'(x) + G(x)$ is satisfied

Proof

$$\begin{aligned} G(x) &= \sum_{n \geq 0} g_n \frac{x^n}{n!} \\ G'(x) &= \frac{\partial}{\partial x} \sum_{n \geq 0} g_n \frac{x^n}{n!} = \sum_{n \geq 0} \frac{\partial}{\partial x} g_n \frac{x^n}{n!} = \sum_{n \geq 0} g_n \frac{nx^{n-1}}{n!} = \sum_{n \geq 0} g_n \frac{nx^{n-1}}{n(n-1)!} = \sum_{n \geq 1} g_n \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{n \geq 0} g_{n+1} \frac{x^n}{n!} \end{aligned}$$

Now, $G(x)$ is the exponential generating function for $g_0, g_1, g_2, \dots$ $G'(x)$ is the exponential generating function for the adjusted sequence $g_1, g_2, g_3, \dots$ Following the same approach, we confirm that $G''(x)$ is the exponential generating function for the sequence $g_2, g_3, g_4, \dots$ Thus, the illustration above is where exponential generating function has the most distinguishing features.

Now, using $g_{n+2} = g_{n+1} + g_n$ as corresponding Fibonacci recurrence, we have

$$G''(x) = \sum_{n \geq 0} g_{n+2} \frac{x^n}{n!} = \sum_{n \geq 0} (g_{n+1} + g_n) \frac{x^n}{n!} = \sum_{n \geq 0} g_{n+1} \frac{x^n}{n!} + \sum_{n \geq 0} g_n \frac{x^n}{n!} = G'(x) + G(x)$$

Therefore, $G''(x) = G'(x) + G(x)$ *proved*.

3.1.2 WILF'S METHOD OF FINDING GENERATING FUNCTION

The Wilf's method of finding generating functions are as follows:

1. Define the generating function as we demonstrated in equation 2.1.
2. Multiply the recurrence relation of both sides by x^n and sum over n for which the recurrence holds.
3. Express explicitly the resulting equation of both sides in terms of the generating function $B(x)$
4. Solve the resulting equation for the unknown generating function $B(x)$
5. If you want an exact formula for the sequence that is defined by the given recurrence relation, then expand $B(x)$ into a power series using any method of your choice (Wilf, 1994).

3.1.3 EULER'S NUMBER TRIANGLE

The numbers were developed by Leonhard Euler in 1755. The Eulerian triangle numbers can also be called Euler's triangle. It is the coefficient of the Euler's polynomials. The number of permutations of $\{1, 2, 3, 4, 5, \dots, k\}$ with n ascents is called the Eulerian numbers and its formula is

$$C(k, n) = \left\langle k \atop n \right\rangle = \sum_{i=0}^n (-1)^i \binom{k+1}{i} (n+1-i)^k \quad (3.1)$$

For example,

$$\begin{aligned}
 C(4,2) &= \sum_{i=0}^2 (-1)^i \binom{4+1}{i} (2+1-i)^4 \\
 &= (-1)^0 \binom{5}{0} (2+1-0)^4 + (-1)^1 \binom{5}{1} (2+1-1)^4 + (-1)^2 \binom{5}{2} (2+1-2)^4 \\
 &= (1)(1)(3)^4 + (-1)(5)(2)^4 + (1)(10)(1)^4 \\
 &= 81 - 80 + 10 = 11
 \end{aligned}$$

The polynomials of Euler popularly called Eulerian polynomials can be presented in the form of generating function of powers

$$g(x) = \frac{x C_d(x)}{(1-x)^{d+1}}$$

$$\text{Where } C_d(x) = \sum_{k=0}^d C(d,k)x^k \text{ for } d \geq 0$$

Thus, the Euler's number triangle is shown in the table below

Table 3.1

$k \setminus n$	0	1	2	3	4	5	6	7	8
0	1								
1	1								
2	1	1							
3	1	4	1						
4	1	11	11	1					
5	1	26	66	26	1				
6	1	57	302	302	57	1			

7	1	120	1191	2416	1191	120	1		
8	1	247	4293	15619	15619	4293	247	1	
9	1	502	14608	88234	156190	88234	14608	502	1

The recursion rule is stated in (3.1) above and $C(0,0) = 1$. The empty squares or cells outside the triangle are regarded as zero value (Weisstein, 2017).

4.1 SPECIAL CASE OF OUR RESEARCH RESULTS

4.1.1 DEVELOP GENERALIZED TECHNIQUES FOR CONSTRUCTING THE GENERATING FUNCTIONS OF THE SEQUENCES OF THE FORM $f_{m+1} = 2(f_m + m^k)$ for all $m \geq 0, k > 0$ and $f_0 = 0$

We started developing the generating functions of the sequences of the form

$$f_{m+1} = 2(f_m + m^k) \text{ for all } m \geq 0, k > 0 \text{ and } f_0 = 0$$

by considering the following sequences;

$$1. f_{m+1} = 2(f_m + m) \text{ for all } m \geq 0 \text{ and } f_0 = 0$$

$$\text{The sequence of } f_{m+1} = 2(f_m + m) = (0, 2, 8, 22, 52, \dots)$$

Therefore, the generating function of the sequence

$$f_{m+1} = 2(f_m + m) = (0, 2, 8, 22, 52, \dots) \text{ is}$$

$$g_1(x) = \sum_{m=0}^{\infty} f_{m+1} x^m = \sum_{m=0}^{\infty} (0, 2, 8, 22, 52, \dots) x^m$$

$$g_1(x) = 2x + 8x^2 + 22x^3 + 52x^4 + \dots$$

Using method of differencing in generating function to get its close form, we have

$$g_1(x) = 2x + 8x^2 + 22x^3 + 52x^4 + \dots$$

$$- \quad xg_1(x) = 2x^2 + 8x^3 + 22x^4 + \dots$$

$$g_1(x)[1 - x] = 2x + 6x^2 + 14x^3 + 30x^4 + \dots$$

$$g_1(x)[1 - x] = 2(x + 3x^2 + 7x^3 + 15x^4 + \dots) \quad (4.1)$$

$$\text{Let } A = x + 3x^2 + 7x^3 + 15x^4 + \dots$$

$$- \quad xA = x^2 + 3x^3 + 7x^4 + \dots$$

$$A[1 - x] = x + 2x^2 + 4x^3 + 8x^4 + \dots$$

$$A[1 - x] = \frac{x}{(1 - 2x)}$$

$$\therefore A = \frac{x}{(1 - 2x)(1 - x)}$$

$$\text{Substitute } A = \frac{x}{(1 - 2x)(1 - x)} \text{ in (4.1), we have}$$

$$g_1(x)[1 - x] = 2 \left(\frac{x}{(1 - 2x)(1 - x)} \right) = \frac{2x}{(1 - 2x)(1 - x)}$$

$$\therefore g_1(x) = \frac{2x}{(1 - 2x)(1 - x)^2}$$

Thus, the generating function for $f_{m+1} = 2(f_m + m)$ for all $m \geq 0$ and $f_0 = 0$ is

$$g_1(x) = \frac{2x}{(1 - 2x)(1 - x)^2}$$

2. $f_{m+1} = 2(f_m + m^2)$ for all $m \geq 0$ and $f_0 = 0$

The sequence of $f_{m+1} = 2(f_m + m^2) = (0, 2, 12, 42, 116, \dots)$

Therefore, the generating function of the sequence

$$f_{m+1} = 2(f_m + m^2) = (0, 2, 12, 42, 116, \dots) \text{ is}$$

$$g_2(x) = \sum_{m=0}^{\infty} f_{m+1}x^m = \sum_{m=0}^{\infty} (0, 2, 12, 42, 116, \dots)x^m$$

$$g_2(x) = 2x + 12x^2 + 42x^3 + 116x^4 + \dots$$

Using method of differencing in generating function to get its close form, we have

$$g_2(x) = 2x + 12x^2 + 42x^3 + 116x^4 + \dots$$

$$- \quad xg_2(x) = 2x^2 + 12x^3 + 42x^4 + \dots$$

$$g_2(x)[1 - x] = 2x + 10x^2 + 30x^3 + 74x^4 + \dots$$

$$g_2(x)[1 - x] = 2(x + 5x^2 + 15x^3 + 37x^4 + \dots) \quad (4.2)$$

$$\text{Let } A = x + 5x^2 + 15x^3 + 37x^4 + \dots$$

$$- \quad xA = x^2 + 5x^3 + 15x^4 + \dots$$

$$A[1 - x] = x + 4x^2 + 10x^3 + 22x^4 + \dots \quad (4.3)$$

$$\text{Let } B = x + 4x^2 + 10x^3 + 22x^4 + \dots$$

$$- \quad xB = x^2 + 4x^3 + 10x^4 + \dots$$

$$B[1 - x] = x + 3x^2 + 6x^3 + 12x^4 + \dots$$

$$B[1 - x] = x + \frac{3x^2}{(1 - 2x)} = \frac{x(1 - 2x) + 3x^2}{(1 - 2x)} = \frac{x - 2x^2 + 3x^2}{(1 - 2x)} = \frac{x^2 + x}{(1 - 2x)}$$

$$\therefore B = \frac{x^2 + x}{(1 - 2x)(1 - x)}$$

Substitute $B = \frac{x^2 + x}{(1 - 2x)(1 - x)}$ in (4.3), we have

$$A[1 - x] = \frac{x^2 + x}{(1 - 2x)(1 - x)}$$

$$\therefore A = \frac{x^2 + x}{(1 - 2x)(1 - x)^2}$$

Substitute $A = \frac{x^2 + x}{(1 - 2x)(1 - x)^2}$ in (4.2), we have

$$g_2(x)[1 - x] = 2 \left(\frac{x^2 + x}{(1 - 2x)(1 - x)^2} \right) = \frac{2(x^2 + x)}{(1 - 2x)(1 - x)^2} = \frac{2x(x + 1)}{(1 - 2x)(1 - x)^2}$$

$$\therefore g_2(x) = \frac{2x(x + 1)}{(1 - 2x)(1 - x)^3}$$

Thus, the generating function for $f_{m+1} = 2(f_m + m^2)$ for all $m \geq 0$ and $f_0 = 0$ is

$$g_2(x) = \frac{2x(x + 1)}{(1 - 2x)(1 - x)^3}$$

3. $f_{m+1} = 2(f_m + m^3)$ for all $m \geq 0$ and $f_0 = 0$

The sequence of $f_{m+1} = 2(f_m + m^3) = (0, 2, 20, 94, 316, 882, 2196, \dots)$

Therefore, the generating function of the sequence

$f_{m+1} = 2(f_m + m^3) = (0, 2, 20, 94, 316, 882, 2196, \dots)$ is

$$g_3(x) = \sum_{m=0}^{\infty} f_{m+1}x^m = \sum_{m=0}^{\infty} (0, 2, 20, 94, 316, 882, 2196, \dots)x^m$$

$$g_3(x) = 2x + 20x^2 + 94x^3 + 316x^4 + 882x^5 + 2196x^6 + \dots$$

Using method of differencing in generating function to get its close form, we have

$$g_3(x) = 2x + 20x^2 + 94x^3 + 316x^4 + 882x^5 + 2196x^6 + \dots$$

$$- \quad xg_3(x) = 2x^2 + 20x^3 + 94x^4 + 316x^5 + 882x^6 + \dots$$

$$g_3(x)[1 - x] = 2x + 18x^2 + 74x^3 + 222x^4 + 566x^5 + 1314x^6 + \dots$$

$$g_3(x)[1 - x] = 2(x + 9x^2 + 37x^3 + 111x^4 + 283x^5 + 657x^6 + \dots) \quad (4.4)$$

$$\text{Let } A = x + 9x^2 + 37x^3 + 111x^4 + 283x^5 + 657x^6 + \dots$$

$$- \quad xA = x^2 + 9x^3 + 37x^4 + 111x^5 + 283x^6 + \dots$$

$$A[1 - x] = x + 8x^2 + 28x^3 + 74x^4 + 172x^5 + 374x^6 + \dots \quad (4.5)$$

$$\text{Let } B = x + 8x^2 + 28x^3 + 74x^4 + 172x^5 + 374x^6 + \dots$$

$$- \quad xB = x^2 + 8x^3 + 28x^4 + 74x^5 + 172x^6 + \dots$$

$$B[1 - x] = x + 7x^2 + 20x^3 + 46x^4 + 98x^5 + 202x^6 + \dots \quad (4.6)$$

$$\text{Let } C = x + 7x^2 + 20x^3 + 46x^4 + 98x^5 + 202x^6 + \dots$$

$$- \quad xC = x^2 + 7x^3 + 20x^4 + 46x^5 + 98x^6 + \dots$$

$$C[1 - x] = x + 6x^2 + 13x^3 + 26x^4 + 52x^5 + 104x^6 + \dots$$

$$C[1 - x] = x + 6x^2 + \frac{13x^3}{(1 - 2x)} = \frac{(x + 6x^2)(1 - 2x) + 13x^3}{(1 - 2x)}$$

$$C[1-x] = \frac{x - 2x^2 + 6x^2 - 12x^3 + 13x^3}{(1-2x)} = \frac{x^3 + 4x^2 + x}{(1-2x)}$$

$$\therefore C = \frac{x^3 + 4x^2 + x}{(1-2x)(1-x)}$$

Substitute $C = \frac{x^3 + 4x^2 + x}{(1-2x)(1-x)}$ in (4.6), we have

$$B[1-x] = \frac{x^3 + 4x^2 + x}{(1-2x)(1-x)}$$

$$\therefore B = \frac{x^3 + 4x^2 + x}{(1-2x)(1-x)^2}$$

Substitute $B = \frac{x^3 + 4x^2 + x}{(1-2x)(1-x)^2}$ in (4.5), we have

$$A[1-x] = \frac{x^3 + 4x^2 + x}{(1-2x)(1-x)^2}$$

$$\therefore A = \frac{x^3 + 4x^2 + x}{(1-2x)(1-x)^3}$$

Substitute $A = \frac{x^3 + 4x^2 + x}{(1-2x)(1-x)^3}$ in (4.4), we have

$$g_3(x)[1-x] = 2 \left(\frac{x^3 + 4x^2 + x}{(1-2x)(1-x)^3} \right) = \frac{2(x^3 + 4x^2 + x)}{(1-2x)(1-x)^3} = \frac{2x(x^2 + 4x + 1)}{(1-2x)(1-x)^3}$$

$$\therefore g_3(x) = \frac{2x(x^2 + 4x + 1)}{(1-2x)(1-x)^4}$$

Thus, the generating function for $f_{m+1} = 2(f_m + m^3)$ for all $m \geq 0$ and $f_0 = 0$ is

$$g_3(x) = \frac{2x(x^2 + 4x + 1)}{(1 - 2x)(1 - x)^4}$$

4. $f_{m+1} = 2(f_m + m^4)$ for all $m \geq 0$ and $f_0 = 0$

The sequence of

$$f_{m+1} = 2(f_m + m^4) = (0, 2, 36, 234, 980, 3210, 9012, \dots)$$

Therefore, the generating function of the sequence

$$f_{m+1} = 2(f_m + m^4) = (0, 2, 36, 234, 980, 3210, 9012, \dots) \text{ is}$$

$$g_4(x) = \sum_{m=0}^{\infty} f_{m+1}x^m = \sum_{m=0}^{\infty} (0, 2, 36, 234, 980, 3210, 9012, \dots)x^m$$

$$g_4(x) = 2x + 36x^2 + 234x^3 + 980x^4 + 3210x^5 + 9012x^6 + \dots$$

Using method of differencing in generating function to get its close form, we have

$$g_4(x) = 2x + 36x^2 + 234x^3 + 980x^4 + 3210x^5 + 9012x^6 + \dots$$

$$- xg_4(x) = 2x^2 + 36x^3 + 234x^4 + 980x^5 + 3210x^6 + \dots$$

$$g_4(x)[1 - x] = 2x + 34x^2 + 198x^3 + 746x^4 + 2230x^5 + 5802x^6 + \dots$$

$$g_4(x)[1 - x] = 2(x + 17x^2 + 99x^3 + 373x^4 + 1115x^5 + 2901x^6 + \dots) \quad (4.7)$$

$$\text{Let } A = x + 17x^2 + 99x^3 + 373x^4 + 1115x^5 + 2901x^6 + \dots$$

$$- xA = x^2 + 17x^3 + 99x^4 + 373x^5 + 1115x^6 + \dots$$

$$A[1 - x] = x + 16x^2 + 82x^3 + 274x^4 + 742x^5 + 1786x^6 + \dots \quad (4.8)$$

$$\text{Let } B = x + 16x^2 + 82x^3 + 274x^4 + 742x^5 + 1786x^6 + \dots$$

$$- \quad xB = x^2 + 16x^3 + 82x^4 + 274x^5 + 742x^6 + \dots$$

$$B[1-x] = x + 15x^2 + 66x^3 + 192x^4 + 468x^5 + 1044x^6 + \dots \quad (4.9)$$

$$\text{Let } C = x + 15x^2 + 66x^3 + 192x^4 + 468x^5 + 1044x^6 + \dots$$

$$- \quad xC = x^2 + 15x^3 + 66x^4 + 192x^5 + 468x^6 + \dots$$

$$C[1-x] = x + 14x^2 + 51x^3 + 126x^4 + 276x^5 + 576x^6 + \dots \quad (4.10)$$

$$\text{Let } D = x + 14x^2 + 51x^3 + 126x^4 + 276x^5 + 576x^6 + \dots$$

$$- \quad xD = x^2 + 14x^3 + 51x^4 + 126x^5 + 276x^6 + \dots$$

$$D[1-x] = x + 13x^2 + 37x^3 + 75x^4 + 150x^5 + 300x^6 + \dots$$

$$D[1-x] = x + 13x^2 + 37x^3 + \frac{75x^4}{(1-2x)} = \frac{(x + 13x^2 + 37x^3)(1-2x) + 75x^4}{(1-2x)}$$

$$D[1-x] = \frac{x - 2x^2 + 13x^2 - 26x^3 + 37x^3 - 74x^4 + 75x^4}{(1-2x)} = \frac{x^4 + 11x^3 + 11x^2 + x}{(1-2x)}$$

$$\therefore D = \frac{x^4 + 11x^3 + 11x^2 + x}{(1-2x)(1-x)}$$

$$\text{Substitute } D = \frac{x^4 + 11x^3 + 11x^2 + x}{(1-2x)(1-x)} \text{ in (4.10), we have}$$

$$C[1-x] = \frac{x^4 + 11x^3 + 11x^2 + x}{(1-2x)(1-x)}$$

$$\therefore C = \frac{x^4 + 11x^3 + 11x^2 + x}{(1-2x)(1-x)^2}$$

$$\text{Substitute } C = \frac{x^4 + 11x^3 + 11x^2 + x}{(1-2x)(1-x)^2} \text{ in (4.9), we have}$$

$$B[1-x] = \frac{x^4 + 11x^3 + 11x^2 + x}{(1-2x)(1-x)^2}$$

$$\therefore B = \frac{x^4 + 11x^3 + 11x^2 + x}{(1-2x)(1-x)^3}$$

Substitute $B = \frac{x^4 + 11x^3 + 11x^2 + x}{(1-2x)(1-x)^3}$ in (4.8), we have

$$A[1-x] = \frac{x^4 + 11x^3 + 11x^2 + x}{(1-2x)(1-x)^3}$$

$$\therefore A = \frac{x^4 + 11x^3 + 11x^2 + x}{(1-2x)(1-x)^4}$$

Substitute $A = \frac{x^4 + 11x^3 + 11x^2 + x}{(1-2x)(1-x)^4}$ in (4.7), we have

$$g_4(x)[1-x] = 2 \left(\frac{x^4 + 11x^3 + 11x^2 + x}{(1-2x)(1-x)^4} \right) = \frac{2(x^4 + 11x^3 + 11x^2 + x)}{(1-2x)(1-x)^4}$$

$$g_4(x)[1-x] = \frac{2x(11x^3 + 11x^2 + 11x + 1)}{(1-2x)(1-x)^4}$$

$$\therefore g_4(x) = \frac{2x(11x^3 + 11x^2 + 11x + 1)}{(1-2x)(1-x)^5}$$

Thus, the generating function for $f_{m+1} = 2(f_m + m^4)$ for all $m \geq 0$ and $f_0 = 0$ is

$$g_4(x) = \frac{2x(11x^3 + 11x^2 + 11x + 1)}{(1-2x)(1-x)^5}$$

⋮

$$f_{m+1} = 2(f_m + m^k) \{ \text{where } f_0 = 0 \text{ and } m = 0, 1, 2, 3, 4, \dots \}$$

The sequence of

$$\begin{aligned} f_{m+1} &= 2(f_m + m^k) \\ &= \{(2 \cdot 0^k), 2(2 \cdot 0^k + 1^k), 2(2 \cdot 1^k + 2^k), 2(4 \cdot 1^k + 2 \cdot 2^k \\ &\quad + 3^k), 2(2(4 \cdot 1^k + 2 \cdot 2^k + 3^k) + 4^k) + \dots\} \end{aligned}$$

$$\begin{aligned} f_{m+1} &= \{(2^1 \cdot 0^k), (2^2 \cdot 0^k + 2 \cdot 1^k), (2^2 \cdot 1^k + 2 \cdot 2^k), (2^3 \cdot 1^k + 2^2 \cdot 2^k + 2 \cdot 3^k), (2^4 \cdot 1^k \\ &\quad + 2^3 \cdot 2^k + 2^2 \cdot 3^k + 2 \cdot 4^k) + (2^5 \cdot 1^k + 2^4 \cdot 2^k + 2^3 \cdot 3^k + 2^2 \cdot 4^k + 2 \cdot 5^k) \\ &\quad + \dots\} \end{aligned}$$

$$f_{m+1} = \sum_{j=1}^{m-1} 2^{m-j} j^k$$

Using method of differencing in generating function to get its close form, we have

$$g_k(x) = \frac{2x[c_k(x)]}{(1-2x)(1-x)^{k+1}}$$

$\therefore$ The generating function for $f_{m+1} = 2(f_m + m^k)$ for all $m \geq 0$ and $f_0 = 0$ is

$$g_k(x) = \frac{2x[c_k(x)]}{(1-2x)(1-x)^{k+1}}$$

where $c_k(x)$ is $(k-1)$ degree polynomial. We shall consider the nature of this polynomial in our result discussion below.

4.1.1.1 DISCUSSION OF THE RESULT

The polynomials formed at the numerators of each of the above illustrated generating functions (i.e. from number 4.1.1 (1) to number 4.1.1 (4)) since the denominator has a clear pattern, can be illustrated using the pyramid below

$$2x(1)$$

$$2x(x + 1)$$

$$2x(x^2 + 4x + 1)$$

$$2x(x^3 + 11x^2 + 11x + 1)$$

The polynomials inside the brackets look exactly like Euler's polynomials which can be gotten from Euler's number as we have illustrated in table 3.1 above. The only difference is the $2x$ in front of each one or line. Therefore, we shall use Euler's number formula to generalize the technique for constructing the generating functions of the sequences of the form $f_{m+1} = 2(f_m + m^k)$ for all $m \geq 0, k > 0$ and $f_0 = 0$ since these polynomials look exactly like Euler's polynomial.

Thus, the generalized technique for constructing the generating functions of the sequences of the form

$$f_{m+1} = 2(f_m + m^k) \text{ for all } m \geq 0, k > 0 \text{ and } f_0 = 0$$

is

$$g_k(x) = \frac{2x[c_k(x)]}{(1-2x)(1-x)^{k+1}} = \frac{2x \sum_{n=0}^{k-1} C(k, n)x^n}{(1-2x)(1-x)^{k+1}}$$

where $C(k, n)$ is called Eulerian numbers and its formula is

$$C(k, n) = \sum_{i=0}^n (-1)^i \binom{k+1}{i} (n+1-i)^k \text{ as we stated in (3.1).}$$

Now, let us test the efficiency of the above generalized technique by allowing $k = 4, i.e. f_{m+1} = 2(f_m + m^4)$. Thus, we obtain the generating functions for the sequence $f_{m+1} = 2(f_m + m^4)$ for all $m \geq 0$ and $f_0 = 0$ using our generalized technique

$$g_k(x) = \frac{2x \sum_{n=0}^{k-1} C(k, n)x^n}{(1-2x)(1-x)^{k+1}}$$

Usually, we consider only the numerator because the denominator has a clear pattern, we have

$$2x \sum_{n=0}^{4-1} C(4, n)x^n = 2x \sum_{n=0}^3 C(4, n)x^n = 2x[C(4,0)x^0 + C(4,1)x^1 + C(4,2)x^2 + C(4,3)x^3]$$

$$\text{where } C(4,0) = \sum_{i=0}^0 (-1)^i \binom{4+1}{i} (0+1-i)^4 = (-1)^0 \binom{5}{0} (0+1-0)^4 = 1(1)(1) = 1$$

$$C(4,1) = \sum_{i=0}^1 (-1)^i \binom{4+1}{i} (1+1-i)^4$$

$$= (-1)^0 \binom{5}{0} (1+1-0)^4 + (-1)^1 \binom{5}{1} (1+1-1)^4$$

$$= 1(1)(2)^4 + (-1)(5)(1)^4 = 16 + (-5) = 16 - 5 = 11$$

$$\therefore C(4,1) = 11$$

$$C(4,2) = \sum_{i=0}^2 (-1)^i \binom{4+1}{i} (2+1-i)^4$$

$$= (-1)^0 \binom{5}{0} (2+1-0)^4 + (-1)^1 \binom{5}{1} (2+1-1)^4 + (-1)^2 \binom{5}{2} (2+1-2)^4$$

$$= (1)(1)(3)^4 + (-1)(5)(2)^4 + (1)(10)(1)^4$$

$$= 81 - 80 + 10 = 11$$

$$\therefore C(4,2) = 11$$

$$C(4,3) = \sum_{i=0}^3 (-1)^i \binom{4+1}{i} (3+1-i)^4$$

$$= (-1)^0 \binom{5}{0} (3+1-0)^4 + (-1)^1 \binom{5}{1} (3+1-1)^4 + (-1)^2 \binom{5}{2} (3+1-2)^4$$

$$+ (-1)^3 \binom{5}{3} (3+1-3)^4$$

$$= (1)(1)(4)^4 + (-1)(5)(3)^4 + (1)(10)(2)^4 + (-1)(10)(1)^4$$

$$= 256 - 405 + 160 - 10 = 416 - 415 = 1$$

$$\therefore C(4,3) = 1$$

$$\therefore 2x \sum_{n=0}^{4-1} C(4,n)x^n = 2x \sum_{n=0}^3 C(4,n)x^n = 2x[C(4,0)x^0 + C(4,1)x^1 + C(4,2)x^2 + C(4,3)x^3]$$

$$= 2x[1x^0 + 11x^1 + 11x^2 + 1x^3]$$

$$= 2x[1 + 11x + 11x^2 + x^3]$$

Therefore, the generating functions for the sequence $f_{m+1} = 2(f_m + m^4)$ for all $m \geq 0$ and $f_0 = 0$ is

$$g_4(x) = \frac{2x \sum_{n=0}^{k-1} C(k,n)x^n}{(1-2x)(1-x)^{k+1}} = \frac{2x(1 + 11x + 11x^2 + x^3)}{(1-2x)(1-x)^{4+1}} = \frac{2x(1 + 11x + 11x^2 + x^3)}{(1-2x)(1-x)^5}$$

Hence, the generating function for the sequence $f_{m+1} = 2(f_m + m^4)$ is

$$g_4(x) = \frac{2x(1 + 11x + 11x^2 + x^3)}{(1 - 2x)(1 - x)^5}$$

This result is the same with the result we have in number 4.1.1 (4).

Similarly, the generalized technique for constructing the generating functions of the sequences of the form

$$f_{m+1} = 2(f_m + (m - 1)^k) \text{ for all } m \geq 1, k > 0 \text{ and } f_1 = 0 \text{ is}$$

$$g_k(x) = \frac{2x^2 \sum_{n=0}^{k-1} C(k, n)x^n}{(1 - 2x)(1 - x)^{k+1}}$$

where $C(k, n)$ is called Eulerian numbers and its formula is

$$C(k, n) = \sum_{i=0}^n (-1)^i \binom{k+1}{i} (n+1-i)^k \text{ as we stated in (3.1).}$$

Now, we can use the above generalized technique to obtain the generating functions of $f_{m+1} = 2(f_m + (m - 1)^4)$. Here, we observe that $k = 4$. Thus, we obtain the generating functions for the sequence $f_{m+1} = 2(f_m + (m - 1)^4)$ for all $m \geq 1$ and $f_1 = 0$ using our generalized technique

$$g_k(x) = \frac{2x^2 \sum_{n=0}^{k-1} C(k, n)x^n}{(1 - 2x)(1 - x)^{k+1}}$$

Usually, we consider only the numerator because the denominator has a clear pattern, we have

$$2x^2 \sum_{n=0}^{4-1} C(4, n)x^n = 2x^2 \sum_{n=0}^3 C(4, n)x^n = 2x^2 [C(4, 0)x^0 + C(4, 1)x^1 + C(4, 2)x^2 + C(4, 3)x^3]$$

$$\text{where } C(4,0) = \sum_{i=0}^0 (-1)^i \binom{4+1}{i} (0+1-i)^4 = (-1)^0 \binom{5}{0} (0+1-0)^4 = 1(1)(1) = 1$$

$$C(4,1) = \sum_{i=0}^1 (-1)^i \binom{4+1}{i} (1+1-i)^4$$

$$= (-1)^0 \binom{5}{0} (1+1-0)^4 + (-1)^1 \binom{5}{1} (1+1-1)^4$$

$$= 1(1)(2)^4 + (-1)(5)(1)^4 = 16 + (-5) = 16 - 5 = 11$$

$$\therefore C(4,1) = 11$$

$$C(4,2) = \sum_{i=0}^2 (-1)^i \binom{4+1}{i} (2+1-i)^4$$

$$= (-1)^0 \binom{5}{0} (2+1-0)^4 + (-1)^1 \binom{5}{1} (2+1-1)^4 + (-1)^2 \binom{5}{2} (2+1-2)^4$$

$$= (1)(1)(3)^4 + (-1)(5)(2)^4 + (1)(10)(1)^4$$

$$= 81 - 80 + 10 = 11$$

$$\therefore C(4,2) = 11$$

$$C(4,3) = \sum_{i=0}^3 (-1)^i \binom{4+1}{i} (3+1-i)^4$$

$$= (-1)^0 \binom{5}{0} (3+1-0)^4 + (-1)^1 \binom{5}{1} (3+1-1)^4 + (-1)^2 \binom{5}{2} (3+1-2)^4$$

$$+ (-1)^3 \binom{5}{3} (3+1-3)^4$$

$$= (1)(1)(4)^4 + (-1)(5)(3)^4 + (1)(10)(2)^4 + (-1)(10)(1)^4$$

$$= 256 - 405 + 160 - 10 = 416 - 415 = 1$$

$$\therefore C(4,3) = 1$$

$$\therefore 2x^2 \sum_{n=0}^{4-1} C(4,n)x^n$$

$$= 2x^2 \sum_{n=0}^3 C(4,n)x^n = 2x^2 [C(4,0)x^0 + C(4,1)x^1 + C(4,2)x^2 + C(4,3)x^3]$$

$$= 2x^2 [1x^0 + 11x^1 + 11x^2 + 1x^3]$$

$$= 2x^2 [1 + 11x + 11x^2 + x^3]$$

Therefore, the generating functions for the sequence $f_{m+1} = 2(f_m + (m-1)^4)$ for all $m \geq 1$ and $f_1 = 0$ is

$$g_4(x) = \frac{2x^2 \sum_{n=0}^{k-1} C(k,n)x^n}{(1-2x)(1-x)^{k+1}} = \frac{2x^2(1 + 11x + 11x^2 + x^3)}{(1-2x)(1-x)^{4+1}} = \frac{2x^2(1 + 11x + 11x^2 + x^3)}{(1-2x)(1-x)^5}$$

Hence, the generating function for the sequence $f_{m+1} = 2(f_m + (m-1)^4)$ for all $m \geq 1$ and $f_1 = 0$ is

$$g_4(x) = \frac{2x^2(1 + 11x + 11x^2 + x^3)}{(1-2x)(1-x)^5}$$

4.1.2 APPLICATIONS OF TECHNIQUE OF GENERATING FUNCTIONS IN SOLVING SOME IMPORTANT PROBLEMS OF RECURRENCE RELATIONS

Example 4.1.2.1 Prove that the generating function

$$y = \sum_{n \geq 0} a_n x^n$$

based on Fibonacci sequence satisfies the ordinary differential equation

$$y'' + by' + cy = 0 \text{ if and only if } b = -(n+2) \text{ and } c = -(n+1)(n+2).$$

Proof

$$\text{Let } y = \sum_{n \geq 0} a_n x^n$$

$$\text{Then, } y' = \sum_{n \geq 0} n a_n x^{n-1} = \sum_{n \geq 1} n a_n x^{n-1} = \sum_{n \geq 0} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n \geq 2} n(n-1) a_n x^{n-2} = \sum_{n \geq 0} (n+1)(n+2) a_{n+2} x^n$$

$(\Rightarrow)$ *if*:

if $b = -(n+2)$ and $c = -(n+1)(n+2)$, then

$$y'' + by' + cy = y'' - (n+2)y' - (n+1)(n+2)y$$

$$= \sum_{n \geq 0} (n+1)(n+2) a_{n+2} x^n - \sum_{n \geq 0} (n+1)(n+2) a_{n+1} x^n - \sum_{n \geq 0} (n+1)(n+2) a_n x^n$$

$$\sum_{n \geq 0} \{(n+1)(n+2) a_{n+2} - (n+1)(n+2) a_{n+1} - (n+1)(n+2) a_n\} x^n$$

$$\sum_{n \geq 0} \{(n+1)(n+2) [a_{n+2} - (a_{n+1} + a_n)]\} x^n$$

$$\sum_{n \geq 0} (n+1)(n+2) \cdot 0 \cdot x^n \quad \text{since } a_{n+2} = a_{n+1} + a_n = 0$$

So that

$$y = \sum_{n \geq 0} a_n x^n$$

Satisfies the ordinary differential equation $y'' + by' + cy = 0$

($\Leftarrow$) **only if:**

$$\text{If } y = \sum_{n \geq 0} a_n x^n$$

satisfies the ordinary differential equation $y'' + by' + cy = 0$

$$\text{then } y'' + by' + cy = 0 \Rightarrow \sum_{n \geq 0} [(n+1)(n+2)a_{n+2} + b(n+1)a_{n+1} + ca_n] x^n = 0$$

So that

$$(n+1)(n+2)a_{n+2} + b(n+1)a_{n+1} + ca_n = 0.$$

But

$$a_{n+2} = a_{n+1} + a_n$$

so,

$$(n+1)(n+2)(a_{n+1} + a_n) + b(n+1)a_{n+1} + ca_n = 0$$

$$[(n+1)(n+2) + b(n+1)]a_{n+1} + [(n+1)(n+2) + c]a_n = 0$$

$$\text{So, } (n+1)(n+2) + b(n+1) = 0 \text{ and } (n+1)(n+2) + c = 0$$

$$\text{Thus, } b = -(n+2) \text{ and } c = -(n+1)(n+2)$$

Hence,

$$y = \sum_{n \geq 0} a_n x^n \text{ satisfies the ordinary differential equation } y'' + by' + cy = 0$$

Only if $b = -(n+2)$ and $c = -(n+1)(n+2)$.

Example 4.1.2.2 Show that the generating function $G(x)$ of the (generating) function

$$g(x) = \sum_{n \geq 0} nx^n$$

is the original (generating) function

$$g(x) = \sum_{n \geq 0} nx^n$$

Proof

$$g(x) = \sum_{n \geq 0} nx^n = \sum_{n \geq 1} nx^n = \sum_{n \geq 0} (n+1)x^{n+1}$$

$$g'(x) = \sum_{n \geq 0} n^2 x^{n-1} = \sum_{n \geq 0} (n+1)^2 x^n$$

$$g''(x) = \sum_{n \geq 0} (n+1)^2 nx^{n-1} = \sum_{n \geq 1} n(n+1)^2 x^{n-1} = \sum_{n \geq 0} (n+1)(n+2)^2 x^n$$

$$g'''(x) = \sum_{n \geq 0} n(n+1)(n+2)^2 x^{n-1} = \sum_{n \geq 0} (n+1)(n+2)(n+3)^2 x^n$$

$$g^{iv}(x) = \sum_{n \geq 0} (n+1)(n+2)(n+3)(n+4)^2 x^n$$

⋮

$$g^{(r)}(x) = \sum_{n \geq 0} \prod_{j=1}^{r-1} (n+j)(n+r)^2 x^n$$

$$g^{(r)}(0) = r \cdot r!$$

$$G(x) = \sum_{n \geq 0} \frac{g^{(r)}(0)}{n!} x^n = \sum_{n \geq 0} \frac{n \cdot n!}{n!} x^n = \sum_{n \geq 0} n x^n = g(x)$$

Example 4.1.2.3 Let $\{g_n\}$ and $\{b_n\}$ be any sequence with the property that g_n is the finite difference of b_n . Show that $g_n = b_{n+1} - 2b_n$ for all $n \geq 0$ using technique of generating functions.

Proof

$$\text{Let } g_k = \Delta b_k = b_{k+1} - b_k \text{ for all } k \geq 0$$

$$\therefore b_{k+1} = g_k + b_k$$

$$\sum_{k=0}^n \binom{n}{k} b_{k+1} x^k = \sum_{k=0}^n \binom{n}{k} g_k x^k + \sum_{k=0}^n \binom{n}{k} b_k x^k$$

$$\sum_{k=1}^{n+1} \binom{n}{k-1} b_k x^k = \sum_{k=0}^n \binom{n}{k} g_k x^k + \sum_{k=0}^n \binom{n}{k} b_k x^k$$

$$\sum_{k=1}^{n+1} \left[\binom{n+1}{k} - \binom{n}{k} \right] b_k x^k = \sum_{k=0}^n \binom{n}{k} g_k x^k + \sum_{k=0}^n \binom{n}{k} b_k x^k$$

$$\sum_{k=1}^{n+1} \binom{n+1}{k} b_k x^k - \sum_{k=1}^{n+1} \binom{n}{k} b_k x^k = \sum_{k=0}^n \binom{n}{k} g_k x^k + \sum_{k=0}^n \binom{n}{k} b_k x^k$$

$$\sum_{k=1}^{n+1} \binom{n+1}{k} b_k x^k - \sum_{k=0}^n \binom{n}{k} b_k x^k = \sum_{k=0}^n \binom{n}{k} g_k x^k + \sum_{k=0}^n \binom{n}{k} b_k x^k$$

$$\sum_{k=0}^{n+1} \binom{n+1}{k} b_k x^k = \sum_{k=0}^n \binom{n}{k} g_k x^k + 2 \sum_{k=0}^n \binom{n}{k} b_k x^k$$

$$b_{n+1} = g_n + 2b_n$$

$$\therefore g_n = b_{n+1} - 2b_n \quad \text{proved.}$$

4.2 CONCLUSION

Generating functions allows us to solve many complicated counting problems with very little effort. It allows us to solve problems of sequence with the method we used while solving problems in algebra. Generating functions provide a bridge between continuous and discrete mathematics. Thus, when given power series to handle, we treat the power series in algebraic way where we have no concern whether the series converges or not. In this paper, we shall developed generalized techniques for constructing the generating functions of sequences of the form $f_{m+1} = 2(f_m + m^k)$ for all $m \geq 0, k > 0$ and $f_0 = 0$ using method of differencing in generating function. Also, we shall apply the technique of generating functions in solving some important problems of recurrence relations.

4.3 RECOMMENDATION

Further research should be done on the generating functions of polylogarithmic functions and zeta functions.

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