

## ALTERNATIVE METHOD OF SOLUTION TO QUARTIC EQUATION

OKOLI, O. C.<sup>1</sup> ORAEKIE, P. A.<sup>2</sup> and OKEKE, N. C.<sup>3</sup>

<sup>1</sup>Department of Mathematics, Chukwuemeka Odumegwu Ojukwu University, Anambra State, Nigeria. Email: [odicomatics@yahoo.com](mailto:odicomatics@yahoo.com), [odico2003@yahoo.com](mailto:odico2003@yahoo.com) Phone:+234-8036941434

<sup>2</sup>Department of Mathematics, Chukwuemeka Odumegwu Ojukwu University, Anambra State, Nigeria. Email: [drsirpauloraekie@gmail.com](mailto:drsirpauloraekie@gmail.com) Phone no: +2347031982483

<sup>3</sup>Department of Mathematics, Chukwuemeka Odumegwu Ojukwu University, Anambra State, Nigeria. Email: [okekennanyelucollins@yahoo.com](mailto:okekennanyelucollins@yahoo.com), [okekecollins4nna@gmail.com](mailto:okekecollins4nna@gmail.com) Phone:+234-7065679297

### ABSTRACT

*Let  $ax^4 + bx^3 + cx^2 + dx + e = 0$  be a general quartic equation in one variable. In line with contributions of several researcher in proffering various methods of solution to the general quartic equation, we wish to present yet another alternative method of solution to the general cubic which is quit novel.*

**Keywords:** Quartic equation, Cubic equation, resolvent equation. Auxiliary equation, seeking solution.

### 1. INTRODUCTION

The subject of solution to polynomial equation can be traced back to the fifteen century mathematician in this field of study. However, literature indicates that polynomial equations have been investigated for centuries and it is still ongoing Okoli (2017). Linear and quadratic polynomial equations were solved in the fifteenth centuries while that of cubic and quartic polynomial equations were comprehensively solved in the sixteenth centuries.

Cubic equation was known since the ancient times, even by the ancient Greeks and the ancient Babylonians and also the ancient Egyptians, who dealt with the problem of doubling the cubic. In the 11<sup>th</sup> century, the famous Mathematician Omar Khayyam discovered a geometrical method to solve cubic equation which could be used to get numerical answer by intersecting a parabola with a circle, and by using this method he found cubic equation can have more than one solution. He could not find algebraic formula for the general but he could only solve cubic equation geometrically.

In 1515 Scipione del Ferro discovered a formula that solved the so called “depressed cubic”. Instead of publishing his solution, Del Ferro kept it a secret until his deathbed telling his student Antonio Fior. Niccolo Fotana known as Tartaglia (1500-1557) wins a mathematical contest by solving many different cubic, and gives his method to Cardan. 1539. Girolamo Cardan (1501-1576) gives complete solution of cubic in his book, The Great Art, or the Rules of Algebra. In that book the Ars magma, Cardano (1545) introduced the technique of substitution by Ludovico Ferrari (1522-1565) that not only solved the cubic and quartic but became indispensable in polynomial algebra.

Since then, several other alternative (analytic) methods for the solution of quartic equation has been proposed by some authors in the literature . As a way of contribution to the existing methods, In this paper we present yet another alternative approach to the solution of a general quartic equation.

## 2. METHODOLOGY

In this section we shall define some important terms as will be needed in the presentation of this work. Let  $\mathbb{N}$  be the set of natural numbers and  $D$  the set of real/complex numbers. Then for any given  $n \in \mathbb{N}$ , a function  $f_n: D \rightarrow D$  is said to be a polynomial of degree  $n$  if there exists a constant  $a_j (j = 0, 1, 2, \dots, n)$  such that  $f$  is given by

$$f_n(x) = \sum_{j=0}^n a_{n-j} x^{n-j}; a_n \neq 0 \text{ --- (1.1)}$$

Equation (1.1) is said to be Depressed if  $a_{n-1} = 0$ , Monic if  $a_n = 1$  and Quartic if  $n = 4$ . So that

$$f_4(x) = \sum_{j=0}^4 a_{4-j} x^{4-j} = a_3 x^3 + a_2 x^2 + a_1 x + a_0; a_3 \neq 0 \text{ --- (1.2)}$$

We seek for the solution  $x$  such that

$$f_4(x) = 0; \Rightarrow a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0; a_3 \neq 0 \text{ --- (1.3)}$$

, then such  $x$  is called the root (zero) of  $f_4$ . For simplicity, we shall henceforth write equation (1.3) as

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \text{ --- (1.4)}$$

Divide (1.4) through by  $a$  and substitute

$$x = y - \frac{b}{4a} \text{ --- (1.5)}$$

we obtain

$$y^4 + py^2 + qy + r = 0 \text{ --- (1.6)}$$

a depressed quartic equation of (1.4). where  $p = \left(\frac{c}{a} - \frac{3b^2}{8a^2}\right)$ ,  $q = \left(\frac{d}{a} - \frac{bc}{2a^2} + \frac{b^3}{8a^3}\right)$  and

$$r = \left(\frac{e}{a} - \frac{bd}{4a^2} + \frac{b^2c}{16a^3} - \frac{3b^4}{256a^4}\right).$$

In order to solve equation (1.4), consequently without loss of generality (1.6), it is most common to seek for zero solution to certain auxiliary function  $g(y; \cdot)$  (with a number of undetermined parameters) which we refer to as ‘seeking solution’ to the depressed quartic equation (1.6). In turn we obtain at least a solution to the quartic equation provided we determine these parameters appropriately.

Kulkarni (2006), solved the quartic equation by seeking for the zero solution of the auxiliary function

$$g(x; b_0, c_0, p) = \frac{(x^2 + b_1 x + b_0)^2 - p^2(x^2 + c_1 x - c_0)^2}{1 - p^2} \text{ --- (1.7)}$$

where  $b_1, b_0, c_1, c_0$  and  $p$  are four unknown parameters to be determine. However, one weakness with this approach is that there are more unknown to be determine than the available equation, hence one need to worry on the particular unknown to be eliminated as remarked by Kulkarni (2006, pg.13) and I quote

“Since there are five unknowns ( $b_1, b_0, c_1, c_0$  and  $p$ ) but only four equations to solve, we assign some convenient value to one unknown, so that all the unknowns can be determined. Let us make  $c_1 = 0 \dots$ ”

However the method of determining or assigning of this convenient value to one unknown again remains unknown, hence it is a matter to worry about. Also, Kulkarni (2009) solved the general quartic equation by considering the zero solution of the auxiliary function

$$g(x; b_1, b_0, c_0) = (x^2 + b_1 x + b_0)^2 - p(x + c_0)^2 \text{ --- (1.8)}$$

where  $b_1, b_0, c_0$  and  $p$  are unknown parameters to be determine. However, there was a minor typographical error associated with one of the equations which was later corrected by Kenneth (2011).

Again, Fathi et al (2012) solved the general quartic equation by considering the zero solution of an auxiliary function with fewer number of undetermined (unknown) parameters

$$g(y; b_0, d_0) = (y^2 + b_0)^2 - (y + d_0)^2 - \dots - (1.9)$$

where  $b_0$  and  $d_0$  are unknown parameters to be determine.

It is in fact worthwhile to mention that, the fewer the unknown parameter the better the approach, hence the need for the choice of parameter(s) to be eliminated does not arise and also if this number of unknown parameter become too few, one runs into the problem of reintroduce additional equation (variable changes) in order to uniquely determine the solution and which may lead to further complexity of solution to the problem, which is the case of Fathi et al (2012, pg. 332-333).

Also, we most point out that Fathi et al (2013), in order to mimic the formula propose by Cardan's for the (seeking) solution of cubic equation which is given by

$$x = \sqrt[3]{U} + \sqrt[3]{V} - \dots - (1.10a)$$

where  $U = u^3$  and  $V = v^3$ ,

they proposed the corresponding (seeking) solution for quartic equation as such

$$x = \sqrt[4]{N_1} + \sqrt[4]{N_2} + \sqrt[4]{N_3} - \dots - (1.10b)$$

where  $N_1 = u^4$ ,  $N_2 = v^4$  and  $N_3 = s^4$ .

Unfortunately, (probably unaware by Fathi et al) the proposed equation (1,10b) is a mere repetition of the exact equation used by Euler as far back as in the year 1733.

Euler proceeded to conjectured that similar formulas existed for higher degrees too, and in particular that the roots of a fifth degree equation could be found as

$$x = \sqrt[5]{A} + \sqrt[5]{B} + \sqrt[5]{C} + \sqrt[5]{D} - \dots - (1.10c)$$

where  $A, B, C, D$  were the roots of some fourth degree resolvent; however, he could not find such a resolvent. Of course, it is believed that Euler's conjecture cannot hold, going by the fact that 100 years later it was proved by Abel and Galois that in general there is no solution by radicals for a fifth degree equation. For more on this, the reader may wish to consult Euler (1773), Euler (1770), Bell (2008) and Sangwin (2006),

At this juncture, we are set to introduce our approach in the theorem that follows in the next section of this work by defining a suitable auxiliary function with probably an ideal number of unknown that determine the solution to the problem uniquely without issue of introducing any variable into the depressed quartic equation as in the case of Fathi et al (2012) nor assigning any convenient value to (eliminate) any of the unknown parameters as in the case of Kulkarni (2006). For the geometric outlook of solution to quartic equation, one may wish to see Nickalls (1993).

### 3. MAIN RESULT

In this section we shall introduce yet another auxiliary function approach to the solution of the general quartic equation (1.5) via equation (1.4). The result of this research work is stated in the theorem that follows.

#### Theorem 3.1

Given the general quartic equation  $ax^4 + bx^3 + cx^2 + dx + e = 0$ , there exist  $u, v, w$  an undetermined parameters such that the zero solution to the auxiliary function  $g(y; u, v, w) = (vy + w)^2 - (y^2 + u)^2$  solves the given quartic equation. Where  $x = y - \frac{b}{4a}$ .

Proof

Observe that

$$\begin{aligned}
 g(y; u, v, w) &= 0; \\
 \Rightarrow (vy + w)^2 - (y^2 + u)^2 &= 0 \text{ --- (1.11)} \\
 \Rightarrow (y^2 + u)^2 &= (vy + w)^2; \Rightarrow y^2 + u = \pm(vy + w); \\
 \Rightarrow y^2 + u &= (vy + w) \text{ or } y^2 + u = -(vy + w); \\
 \Rightarrow y^2 - vy + (u - w) &= 0 \text{ or } y^2 + vy + (u + w) = 0;
 \end{aligned}$$

$$\Rightarrow y_1 = \frac{v - \sqrt{v^2 - 4(u - w)}}{2} \text{ or } y_2 = \frac{v + \sqrt{v^2 - 4(u - w)}}{2} \text{ --- (1.12a)}$$

$$\Rightarrow y_3 = \frac{-v - \sqrt{v^2 - 4(u + w)}}{2} \text{ or } y_4 = \frac{-v + \sqrt{v^2 - 4(u + w)}}{2} \text{ --- (1.12b)}$$

Constitute the seeking solution to the quartic equation (1.6). Now expanding eq(1.11) we obtain

$$y^4 + (2u - v^2)y^2 - 2wvy + u^2 - w^2 = 0 \text{ --- (1.13)}$$

Now, comparing the coefficients in the two equations (1.6) and (1.13) we obtained

$$p = (2u - v^2) \Rightarrow u = \frac{p + v^2}{2} \text{ --- (1.13)}$$

$$q = -2wv \Rightarrow w = \frac{-q}{2v} \text{ --- (1.14)}$$

$$r = (u^2 - w^2) \Rightarrow r = \left(\frac{p + v^2}{2}\right)^2 - \left(\frac{-q}{2v}\right)^2 \text{ --- (1.15)}$$

We expand (1.15) to have

$$v^6 + 2pv^4 + (p^2 - 4r)v^2 - q^2 = 0 \text{ --- (1.17)}$$

Which is the resolvent equation associated with the quartic equation

Let

$$v^2 = z \text{ --- (1.18)}$$

Then by substituting into (1.17) we have

$$z^3 + 2pz^2 + (p^2 - 4r)z - q^2 = 0$$

Thus, using any standard method for solving cubic equation we obtain at least one solution in  $z$ , consequently  $v$  using the relating equation (1.18).

Now, recall from (1.13) and (1.14) that  $u = \frac{p+v^2}{2}$  and  $w = \frac{-q}{2v}$  which implies that

$$u + w = \frac{v(v^2 + p) - q}{2v} \text{ and } u - w = \frac{v(v^2 + p) + q}{2v}$$

Then the above seeking solution to (1.6) becomes

$$\begin{aligned}
 y_1 &= \frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) + q}{2v}} \text{ or } y_2 = \frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) + q}{2v}} \\
 y_3 &= -\frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) - q}{2v}} \text{ or } y_4 = -\frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) - q}{2v}}
 \end{aligned}$$

Consequently the solution to (1.4) becomes

$$x_1 = \frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) + q}{2v}} - \frac{b}{4a} \text{ or } x_2 = \frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) + q}{2v}} - \frac{b}{4a}$$

$$x_3 = -\frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) - q}{2v}} - \frac{b}{4a} \text{ or } x_4 = -\frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) - q}{2v}} - \frac{b}{4a}$$

This completes the proof.

**Example** find the roots of  $x^4 + 4x^3 + 33x^2 + (58 - 14i)x + (148 - 14i) = 0$

**Solution**

Reduce above equation into depressed quartic equation by substituting  $x = y - \frac{b}{4a} \Rightarrow x = y - \frac{4}{4 \times 1} \Rightarrow x = y - 1$  to obtain

$$y^4 + 27y^2 - 14iy + 120 = 0$$

Using resolvent cubic equation; that is

$$z^3 + 2z^2p + (p^2 - 4r)z - q^2 = 0$$

where  $p = 27, q = -14i, r = 120$  substitute this values we have

$$z^3 + 54z^2 + 249z + 196 = 0$$

We can use any standard method to obtain the first root of the resolvent cubic equation, hence we see that  $z = -1$  is one of such root. Since  $v^2 = z \Rightarrow v = \pm i$ .

Using  $v = i, a = 1, b = 4, p = 27, q = -14i$  we see that the roots of the equation are

$$x_1 = \frac{-b}{4a} + \frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) + q}{2v}} = \frac{-4}{4} + \frac{i}{2} + \sqrt{\frac{i^2}{4} - \frac{i(i^2 + 27) + (-14i)}{2i}} = -1 + 3i$$

$$x_2 = \frac{-b}{4a} + \frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) + q}{2v}} = \frac{-4}{4} + \frac{i}{2} - \sqrt{\frac{i^2}{4} - \frac{i(i^2 + 27) + (-14i)}{2i}} = -1 - 2i$$

$$x_3 = \frac{-b}{4a} - \frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) - q}{2v}} = \frac{-4}{4} - \frac{i}{2} + \sqrt{\frac{i^2}{4} - \frac{i(i^2 + 27) - (-14i)}{2i}} = -1 + 4i$$

$$x_4 = \frac{-b}{4a} - \frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) - q}{2v}} = \frac{-4}{4} - \frac{i}{2} - \sqrt{\frac{i^2}{4} - \frac{i(i^2 + 27) - (-14i)}{2i}} = -1 - 5i$$

Note that if we use  $v = -i$  we obtain the same values for the roots  $x_j$  ( $j = 1, 2, 3, 4$ ) above.

**Example** find the roots of  $y^4 - y^2 + 2y + 2 = 0$

**Solution** Since the equation is a depressed quartic equation we solve using resolvent cubic equation that is

$$z^3 + 2z^2p + (p^2 - 4r)z - q^2 = 0$$

Observe that  $p = -1, q = 2, r = 2$ , Substituting this values we have

$$z^3 - 2z^2 - 7z - 4 = 0$$

We can use any standard method to obtain the first root of the resolvent cubic equation, hence we see that  $z = 4$  is one of such root. Since  $v^2 = z \Rightarrow v = \pm 2$ .

Using  $v = 2, a = 1, b = 0, p = -1, q = 2$ , then we observe the roots are

$$x_1 = \frac{-b}{4a} + \frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) + q}{2v}} = \frac{0}{4} + \frac{2}{2} + \sqrt{\frac{2^2}{4} - \frac{2(2^2 - 1) + 2}{2 \times 2}} = 1 + i$$

$$x_2 = \frac{-b}{4a} + \frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) + q}{2v}} = \frac{0}{4} + \frac{2}{2} - \sqrt{\frac{2^2}{4} - \frac{2(2^2 - 1) + 2}{2 \times 2}} = 1 - i$$

$$x_3 = \frac{-b}{4a} - \frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) - q}{2v}} = \frac{0}{4} - \frac{2}{2} + \sqrt{\frac{2^2}{4} - \frac{2(2^2 - 1) - 2}{2 \times 2}} = -1$$

$$x_4 = \frac{-b}{4a} - \frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v(v^2 + p) - q}{2v}} = \frac{0}{4} - \frac{2}{2} + \sqrt{\frac{2^2}{4} - \frac{2(2^2 - 1) - 2}{2 \times 2}} = -1$$

#### 4. CONCLUSION

So far we have demonstrated with telling examples the validity and simplicity of our result and method when compared with the methods employed by some authors in the literature.

#### 5. ACKNOWLEDGEMENT

We wish to thank the staff and colleagues of Chukwuemeka Odumegwu Ojukwu University, Anambra State, Nigeria, in particular, the Department of Mathematics for their supportive attitude and encouragement during the course of this research work.

#### REFERENCE

- Amir Fathi, Pooya Mobadersany, Rahim Fathi, (2012); A Simple Method to Solve Quartic Equations, Australian Journal of Basic and Applied Sciences, 6(6): 331-33
- Amir Fathi, Nastaran Sharifan. (2013); A Classic New Method to Solve Quartic Equations, Applied and Computational Mathematics. Vol. 2, No.2, pp. 24-27.
- Bell J (2008). A conjecture on the forms of the roots of equations. arXiv:0806.1927v1 [math.HO]. <http://arxiv.org/abs/0806.1927> [An English translation of Euler's De formis radicum aequationum cujusque ordinis conjectatio (E30)].
- C. R. Sangwin (Ed.). Euler's Elements of Algebra. Tarquin Publications, St Albans, UK (2006). [English translation of Euler 1770 (E387)]
- Nickalls, R. W. D. (1993); A new approach to solving the cubic: Cardan's solution revealed. Math. Gazette 77, no. 480, 354–359.
- Cardano, G. (1545); Artis magnae, sive de regulis algebraicis. Lib. unus. Qui and totius operis de arithmetica, quod opus perfectum inscripsit, est in ordine decimus. (Ars Magna.) Johann Petreius, Nuremberg. English transl.: The Great Art, or The Rules of Algebra. Translated and edited by T. R. Witmer, MIT Press, Cambridge, Mass., 1968.
- Kenneth V. Cartwright (2011); Correction To "A New Method For Solving Quartics"
- Sutra: International Journal Of Mathematics Science Education @Technomathematics Research Foundation Vol. 4, No. 2, Pp. 30 – 31.
- Kulkarni, R. G. (2009); A new method for solving quartics, Sutra: International Journal of Mathematical Science Education, 2, pp. 24-26.
- R. G. Kulkarni, (2006) Unified method for solving general polynomial equations of degree less than five, Alabama Journal of Mathematics, 30, 1-18.
- Euler L., (1733) De formis radicum aequationum cujusque ordinis conjectatio. Commentarii academiae scientiarum imperialis Petropolitanae; 6 (pub. 1738), pp. 216–231 = Opera Omnia, Series 1 (Pure mathematics), vol. 6 (Theory of equations), pp. 1–19. [Euler Archive, E30 (Latin):<http://www.eulerarchive.org/pages/E030.html>].
- Euler L., Vollständige Anleitung zur Algebra (Elements of algebra), 2 vols, Royal Academy of Sciences, St. Petersburg (1770). [Euler Archive, E387 (English): <http://www.eulerarchive.org/docs/originals/E387e.P1S4.pdf>].

- Nickalls R.W.D., (2012); The quartic equation: alignment with an equivalent tetrahedron, Mathematical Gazette, 96, 49–55;  
<http://www.nickalls.org/dick/papers/math/tetrahedron2012.pdf>.
- Okoli O.C. (2017). A Unify Method for Solving Quadratic Equations. Tropical Journal of Applied Natural Sciences, 2(1): 78-83. Doi:  
<https://doi.org/10.25240/TJANS.2017.2.1.13>